The Philippine Statistician
1991, Volume 40, Nos 1-2 & 3-4
Pages 19-31

WEAK LAWS OF LARGE NUMBER FOR L^1-MIXINGALES

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Abstract

This paper present L^1-convergence results and weak laws of large numbers for L^1-mixingales. Using the approach of McLeish (1975a), results on L^1-convergence are obtained without imposing the requirement used in Andrews (1988) that the random variables in the sequence be uniformly integrable.

1. Introduction

McLeish (1975a, 1975b, 1977) defined a class of dependent random variables called mixingales, and developed the asymptotic theory for these dependent sequences. Mixingales include broad classes of dependent processes such as m-dependent sequences, mixing sequences and ARMA processes (in Section 2 below). Applications of mixingales can be found in Gallant (1987) and Gallant and White (1988).

McLeish (1975a) establishes SLLNs under the assumption that the mixingale numbers sequence decays to zero at a certain rate. Using a weaker moment assumption, Andrews (1988) establishes WLLN without imposing a rate condition on the mixingale number sequence. However, he imposes a uniform integrability condition.

In this paper, WLLNs for L^1-mixingales are established without imposing the uniform integrability condition. However, the infite sum of the mixingale numbers is assumed to go to 0 as the number of observations increases. Our approach is to use a variation of McLeish's (1975) representation of integrable random variables. It can be shown that Andrews' (1988) results can be accomodated under this approach.

The rest of this paper is organized as follows. In Section 2, the definition and examples of L^1-mixingales are presented. In Section 3, WLLNs for L^1-mixingale are established. Relationships between our results and previous results, especially those of Andrews (1988) and Mcleish (1975a), are given in Section 4. The proofs are given in Section 5.

2. L^1-Mixingales

Let (Ω, F, P) denote a probability space. Let $(X_1:i\geq 1)$ be a sequence of random variables on (Ω , F, P). Let $\{F_i:i=\ldots,0,1,\ldots\}$ be any nondecreasing sequence of sub-σ-fields of F. Let $E_jX_i = E(X_i|F_j)$ denote the conditional expectation of X_i given F_j and let $\|\cdot\|_p$ denote the $L^p(P)$ norm, i.e., $\|X_i\|_p=(E|X_i|^P)^{1/P}$.

DEFINITION 1. The sequence (X_i, F_i) is an L^P-mixingale if there exist non-negative constants $(c_i: i\geq 1)$ and (Ψ_m: m≥0) such that for all $i \geq 1$ and $m \geq 0$ we have

(a) $\|E_{i-m}X_i\|_p \leq c_i \Psi_m$ and

(b) $\|X_i = E_{i+m} X_i\|_p \leq c_i \Psi_{m+1}$.

The term mixingale as originally defined in McLeish (1975) is an L^2-mixingale in the context of this definition. Andrews (1988), on the other hand, requires that $\Psi_m \to 0$ as $m \to \infty$ in defining L^1-mixingale.

The following are examples of L^1-mixingales.

(1) A martingale difference array $\{X_1, F_i:1 \leq i \leq n\}$ is an L^1-mixingale. Take $\Psi_m = 0$ for $m \geq 1$, $c_i = \|X_i\|_1$ and set $F_i = \{\varphi,\Omega\}$ for $i \leq 0$ and $F_i = F$ for $i > n$.

(2) An m-dependent sequence of random variables $\{X_i: i \geq 1\}$ is an L^1-mixingale with $\Psi_k = 0$ for $k > m$ and $c_i = \|X_i\|_1$ if one takes $F_i = \sigma(X_1,\ldots,X_i)$ for $1 \leq i \leq n_1$, $F_i = \{\varphi, \Omega\}$ for $i \leq 0$, and $F_i = F$ for $i > n$.

(3) Suppose $X_i = \sum_{j=-\infty}^{\infty} a_{ij}\epsilon_{i-j}$ for $i \geq 1$, where $\{\epsilon_j, F_j : -\infty<j<\infty\}$ is a sequence of martingale difference innovation random variables and corresponding σ-fields and $\{a_{ij} : -\infty < j < \infty,\ i \geq 1\}$ is a sequence of constants. If $\{\epsilon_j\}$ are L^r bounded for some $r > 1$, i.e., $E|\epsilon_j|^r \leq K < \infty$, and $\sum_{j=-\infty}^{\infty} \sup_{i\geq 1} |a_{ij}| < \infty$, then $\{x_i, f_j\}$ is L^1-mixingale with $c_i = \sup_k \|\epsilon_k\|_1$ for $i \geq 1$ and $\Psi_m = \| \sum_{j=m}^{\infty} \sup a_{ij}\|_1$.

3. Weak Laws of Large Numbers

Let $\{X_i\}$ be an L^1-mixingale with associated constants $\{c_i\}$ and $\{\Psi_m\}$. Let

$$(1) \qquad S_n = \sum_{i=1}^{n} X_i$$

Throughout this paper we assume that $EX_i = 0$.

The following proposition is useful in obtaining the upper bound for the sum in (1) when the X_i's satisfy the mixingale condition. The proposition is a variation of McLeish's 1975 result.

PROPOSITION 1. Let $\{X_i\}$ be any sequence of integrable random variables, F_n any nondecreasing sequence of σ-algebras such that ${}_{-\infty}X_i \equiv X_i - E_\infty X_i = 0$ a.s. for all i. Then the partial sum S_n in (1) has representation as an infinite sum of integrable random variables:

$$(2) \qquad S_n = \sum_{k=M}^{\infty} [Y_{n,k} + Z_{n,k}] + U_{n,M}, \qquad M \in I^+$$

where

$$(3) \qquad Y_{n,k} = \sum_{i=1}^{n} [E_{i+k} X_i - E_{i+k-1} X_i]$$

$$(4) \qquad Z_{n,k} = \sum_{i=1}^{n} [E_{i-k} X_i - E_{i-k-1} X_i]$$

(6) $$U_{n,M} = \sum_{i=1}^{n} [E_{i+M-1} X_i - E_{i-M} X_i].$$ ▪

Proposition 1 goes a step beyond McLeish's result by decomposing the sum S_n into more terms. We obtain a finer decomposition to facilitate the bounding of the function of S_n in terms of Ψ_m.

Now we establish the upper bound for a function of $|S_n|$. This is the analog of the result in McLeish (1975a) on $E(\max_{j\le n} S_n^2)$.

THEOREM 1. Let (X_i, F_i) be an integrable L^1-mixingale. If $\sum_{k=0}^{\infty} \Psi_k < \infty$, then there exists a B depending on $\{\Psi_m\}$ such that

(7) $$\left(E \max_{j\le n} |S_j|\right) \le B \left[\sum_{i=1}^{n} c_i \right].$$

In particular, $B = 6 \sum_{k=M}^{\infty} \Psi_k + (\Psi_0 + \Psi_1)$. ▪

To obtain his result on L^2-mixingales, McLeish assumes that the sequence $\{\Psi m\}$ is eventually bounded and specifies the rate at which it attains the upper bound. Theorem 1, on the other hand, assumes the finiteness of B.

COROLLARY 1. Suppose the sequence $\{X_i, F_i\}$ is an integrable L^1-mixingale. If $\lim_{n\to\infty} \sum_{i=1}^{n} c_i < \infty$ and $\sum_{k=M}^{\infty} \Psi_k < \infty$, then

(8) $$|S_n| = \left|\sum_{i=1}^{n} X_i\right| \text{ converges in } L_1.$$ ▪

CORROLARY 2. Suppose the sequence $\{X_i, F_i\}$ is an integrable L^1-mixingale. If $\lim_{n\to\infty} \sum_{i=1}^{n} c_i < \infty$ and

(9) $\lim_{n \to \infty} n^{-1} \sum_{k=M}^{\infty} \Psi_k \to 0$ as $n \to \infty$.

then

(10) $E|n^{-1} S_n| = E|n^{-1} \sum_{i=1}^{n} X_i| \to 0$ as $n \to \infty$

and in consequence $n^{-1} S_n \xrightarrow[L^1]{} 0$ as $n \to \infty$. Therefore,

$n^{-1} S_n \to_p 0$ as $n \to \infty$.

We note that in his 1975a paper, McLeish provides an a.s convergence result under a stronger moment assumption, namely the boundedness of the first moment. Corollary 1, on the other hand, is a result on L^1-convergence under a weaker moment assumption.

Notice the flexibility of Theorem 1 and, specially Corollary 2. Andrews (1988) did not impose a rate of decay to zero on the L^1-mixingale numbers (Ψ_M), but instead he imposed the condition that $\Psi m \to 0$ as $m \to \infty$. This contrasts with many WLLNs in which the constants that index the temporal dependence, such as $\varphi(.)$, $\rho(.)$, or $\alpha(.)$ mixing numbers must converge to zero at a particular rate (Andrews, 1988). In our case, Andrews' requirement regarding the mixingale numbers Ψ_m's is taken care of by Corollary 2. This can be seen from Toeplitz' Lemma which states that if a sequence of real numbers $\{a_n : n \geq 1\}$ satisfies $a_n \to a$ as $n \to \infty$, then $n^{-1} \sum_{k=0}^{n} a_k \to a$ as $n \to \infty$. In this case, take $a_m = \Psi_m$ and apply Toeplitz' Lemma, noting that $n^{-1} \sum_{k=0}^{\infty} a_k$ and $n^{-1} \sum_{k=M}^{\infty} a_k$ converge together.

We remark here that, since $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ are either both convergent or both divergent, (7) can be replaced by $\sum_{k=0}^{\infty} a_k < \infty$. However, the condition (7) is preferred because it lends to easy generalization to other conditions such as those in Andrews (1988).

4. RELATIONSHIPS WITH PREVIOUS RESULTS

The condition $\lim_{n\to\infty} \sum_{i=1}^{n} c_i < \infty$ might be too stringent in some situations. For example, suppose (X_i) is such that $\sup_{i\geq 1} \|X_i\|_1 < \infty$ and we take $c_i = \|X_i\|_1$, $i = 1,2,\ldots$. In this case, the bound will be too large such that the convergence to 0 is not attained. However, strengthening some of the conditions in Corollary 2 will ensure L^1-convergence.

In the following theorem, we strengthen the condition on $\{\Psi_m: m \geq 1\}$, that is to say, we specify its form. The theorem illustrates how our results are related to Andrews' result.

THEOREM 2. Suppose the sequence $\{X_i, F_i\}$ is an L^1-mixingale such that $E \sup_{k\geq 1} |X_k| < \infty$. Assume further that there exists a non-increasing function $f(n,k)$ such that and for all $k < n$,

(11) $\Psi_k = f(n,k)\ \Psi_n$

where for a fixed k, $f(n,k) \to c < \infty$ as $n \to \infty$. Let $c_i = \|X_i\|_1$. If

(12) $\Psi_k \to 0$ as $k \to \infty$,

then as $n \to \infty$, $n^{-1} S_n$ converges to 0 in L_1 and, in consequence $n^{-1} S_n \to_p 0$ as $n \to \infty$. ▪

Theorem 2 actually illustrates the relationship between our result with that of Andrews (1988). An example of a function f(.) defined in Theorem 2 is given below.

The condition $\lim_{n\to\infty} \sum_{i=1}^{n} c_i < \infty$ can also be weakened to $\limsup_{n\to\infty} n^{-1} \sum_{i=1}^{n} c_i < \infty$ when the second moment exists. This is given in the following theorem.

THEOREM 3. Let (X_i, F_i) be an integrable L^2-mixingale. If $\sup_{k\geq 1} \|X_k\|_2 < \infty$, and if $n^{-\delta} \sum_{k=0}^{\infty} \Psi_k < \infty$, $0 < \delta < 1/2$, then $n^{-1} S_n$ converges in L^2 to 0 as $n \to \infty$. ▪

EXAMPLES

We will show an example of the function f(.) defined in Theorem 2. We need the following definition of size. This definition is found in McLeish, 1975a.

DEFINITION 2. A sequence $\{\Psi_m\}$ is of size -p if there exists a positive sequence $\{L(n)\}$ such that:

(a) $\Sigma_n 1/nL(n) < \infty$,

(b) $L(n) - L(n-1) = O(L(n)/n)$,

(c) $L(n)$ is eventually nondecreasing,

(d) $\Psi_n = O(1/(n^{1/2}L(n))^{2p})$

Suppose that $\{\Psi_m\}$ is a mixingale number sequence of size -p. By condition (d), there exists a $\triangle_2$ such that $\Psi n \le \triangle_2/(n^{1/2}L(n))^{2p}$. Therefore, assuming we have the inequality, we obtain by using (b) and (d),

$$(13) \quad \Psi_{n+1} = \left[\{n/(n+1)\}^{1/2}\{(L(n)/L(n+1)\}\right]^{2p} \Psi_n \, .$$

By condition (a) the term L(n)/L(n+1) is bounded. Therefore, the expression in the brackets [.] in (13) is also bounded.

Now, solving (13) recursively, we obtain

$$\begin{aligned}
(14)\ \Psi_{n+1} &= \left[\{n^{1/2}/(n+1)^{1/2}\}\{L(n)/L(n+1)\}\right]^{2p} \Psi_n \\
&= \left[\{n/(n+1)\}^{1/2}\{L(n)/L(n+1)\}\right]^{2p} \\
&\qquad \times \left[\{(n-1)/n\}^{1/2}\{L(n-1)/L(n)\}\right]^{2p} \Psi_{n-1} \\
&\ \ \vdots \\
&= \left[\{(n-k)/(n+1)\}^{1/2}\{(L(n-k)/L(n+1)\}\right]^{2p} \Psi_{n-k} \\
&= \left[\{m/(m+k+1)\}^{1/2}\{L(m)/L(m+k+1)\}\right]^{2p} \Psi_m \\
&= f(m,k)\, \Psi_m,
\end{aligned}$$

where $f(m,k) = \left[\{m/(m+k+1)^{1/2}\}\{L(m)/L(m+k+1)\}\right]^{2p}$. Note that $f(m,k)$ in (14) satisfies the conditions of $f(n,k)$ in Theorem 2. This proves the assertion. ▫

5. PROOFS

PROOF OF PROPOSITION 1. We follow McLeish (1975). Write

$$X_i' = \sum_{k=-m}^{n} (E_{i+k}X_i - E_{i+k-1}X_i)$$

Then,

(15)

$$S_n = \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} (E_{i+k}\, X_i - E_{i+k-1}\, X_i).$$

For fixed $M \in I^+$,

$$S_n = \sum_{i=1}^{n} \sum_{k=M}^{\infty} (E_{i+k}\, X_i - E_{i+k-1}\, X_i)$$

$$+ \sum_{i=1}^{n} \sum_{k=-\infty}^{-M} (E_{i+k}\, X_i - E_{i+k-1}\, X_i)$$

$$+ \sum_{i=1}^{n} (E_{i+M-1}\, X_i - E_{i-M}X_i)$$

$$= \sum_{k=M}^{\infty} \left[Y_{n,k} + Z_{n,k}\right] + U_{n,M}$$

which completes the proof. ▫

PROOF OF THEOREM 1. From Proposition 1, we have

(16)
$$\max_{j \le n} |S_j| \le \sum_{k=M}^{\infty} \sum_{i=1}^{n} (|E_{i+k} X_i - X_i| + |X_i - E_{i+k-1} X_i|)$$
$$+ \sum_{k=M}^{\infty} \sum_{i=1}^{n} (|E_{i-k} X_i| + |E_{i-k-1} X_i|)$$
$$+ \sum_{i=1}^{n} (|E_{i+M-1} X_i| + |E_{i-M} X_i|).$$

Therefore, we have

(17)
$$(E \max_{j \le n} |S_j|) \le \sum_{k=M}^{\infty} \sum_{i=1}^{n} (c_i \Psi_{k+1} + c_i \Psi_k + c_i \Psi_k, + c_i \Psi_{k+1})$$
$$+ \sum_{i=1}^{n} (c_i \Psi_M + c_i (\Psi_0 + \Psi_1) + c_i \Psi_M)$$
$$\le \left[6 \sum_{k=M}^{\infty} \Psi_k + \Psi_0 + \Psi_1 \right] \left[\sum_{i=1}^{n} c_i \right]$$

The last inequality holds because the c_i's and Ψ_1's are positive constants. •

PROOF OF THEOREM 2. To prove the assertion, we need to choose an M in Theorem 1, such that with the chosen M,

(18)
$$n^{-1} \left[6 \sum_{k=M}^{\infty} \Psi_k + \Psi_0 + \Psi_1 \right] \left[\sum_{i=1}^{n} c_i \right] \to 0 \quad \text{as } n \Rightarrow \infty.$$

By assumption (12), we can choose an $M \epsilon I^+$ sufficiently large, but fixed, such that for a given ϵ,

$$(19) \qquad 6 \sum_{k=M}^{\infty} \Psi_k = 6 \left(\sum_{k=M}^{\ell-1} \Psi_k + \sum_{k=\ell}^{\infty} \Psi_k \right) < \epsilon, \ M < \ell .$$

By assumption (11), we can express Ψ_0 and Ψ_1 as functions of Ψ_ℓ, namely,

$$(20) \qquad \Psi_0 = f(\ell,0)\Psi_\ell \quad \text{and} \quad \Psi_1 = f(\ell,1)\Psi_\ell .$$

Substituting (20) in (19), $c_i = \|X_i\|_1$ and using the assumption that $\|X_i\|_1 \le E \sup_{k\ge1} |X_k| < \infty$,

$$n^{-1} \left[6 \sum_{k=M}^{\infty} \Psi_k + \Psi_0 + \Psi_1 \right] \left[\sum_{i=1}^{n} c_i \right]$$

$$= n^{-1} \left[6\left(\sum_{k=M}^{\ell-1} \Psi_k + \sum_{k=\ell}^{\infty} \Psi_k \right) + f(\ell,0)\Psi_\ell + f(\ell,1)\Psi_\ell \right]$$

$$\times \left[\sum_{i=1}^{n} \|X_i\|_1 \right]$$

$$\le \left[6\left(\sum_{k=M}^{\ell-1} \Psi_k + \sum_{k=\ell}^{\infty} \Psi_k \right) + f(\ell,0)\Psi_\ell + f(\ell,1)\Psi_\ell \right]$$

$$\times \left[n^{-1} \sum_{i=1}^{n} E \sup_{k\ge1} |X_k| \right]$$

$$= \left[6\left(\sum_{k=M}^{\ell-1} \Psi_k + \sum_{k=\ell}^{\infty} \Psi_k \right) + f(\ell,0)\Psi_\ell + f(\ell,1)\Psi_\ell \right]$$

$$\times \left[E \sup_{k\ge1} |X_k|_1 \right]$$

Since, by assumption (11), $\Psi_\ell \to 0$ as $\ell \to \infty$, the RHS is less than ϵ', for sufficiently large ℓ. And, hence, the RHS goes to 0 for a given suffiently large ℓ. Therefore, $n^{-1}S_n$ converges in L_1 to 0, and the assertion follows. ▫

PROOF OF THEOREM 3. We follow Andrews (1988). Note that

$$(21) \qquad E(n^{-1}S_n)^2 \le E(2 \; n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{i} X_i X_j)$$

$$\le 2 \; n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{i} |EX_i X_j|$$

Now,

$$(22) \quad |EX_iX_j| \le |EX_i(X_j - E_{j+s}X_j| + |EX_iE_{j+s}X_j|$$

$$= |EX_i(X_j - E_{j+s}X_j)| + |EE_{j+s}(X_iE_{j+s}X_j)|$$

$$\le \|X_i\|_2 \|X_j - E_{j+s}X_j\|_2$$

$$+ \|E_{j+s}X_i\|_2 \|E_{j+s}X_j\|_2$$

$$= (\Psi_0 + \Psi_1) \; \Psi_{s+1} c_i c_j$$

$$+ (\Psi_{s+1} + \Psi_0 + \Psi_1) \Psi_{(i-j-s)} c_i c_j$$

Subtituting (22) in (21) with $s = [(1-j)/2]$,

$$(23) \qquad E(n^{-1}S_n)^2$$

$$\le 4 \; n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{i} (\Psi_0 + \Psi_1) \; \Psi_{[i-j)/2]} c_i c_j$$

$$+ 2 \; n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{i} \Psi^2_{[(i-j)/2]} c_i c_j .$$

Let $c_i = \|X_i\|_2$. Since $c_i \le \sup_{k \ge 1} \|X_i\|_2$, $i = 1,2,\ldots n$,

$$E(n^{-1}S_n)^2 \le \sup_{k\ge1} \|X_k\|_2^2 \; 4 \; n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{i} (\Psi_0 + \Psi_1) \; \Psi_{[(i-j)/2]} .$$

$$+ \sup_{k\ge1} \|X_k\|_2^2 \; 2 \; n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{i} \Psi^2_{[i-j)/2]}$$

$$\leq \sup_{k\geq 1} \|X_k\|_2^2 \; 8\, n^{-1} \sum_{u=1}^{[n/2]} (\Psi_0 + \Psi_1)\, \Psi_u$$

$$+ \sup_{k\geq 1} \|X_k\|_2^2 \; 4\, n^{-1} \sum_{u=1}^{[n/2]} (\Psi_0 + \Psi_1)\, \Psi_u^2 ,$$

which converges to 0 as n -> ∞. This completes the proof. ▪

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